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Balancing Annuities and Tontines: A Study of Temporal Risk Aversion in Retirement Planning

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This article investigates individual demand for retirement income products in a setting where both traditional life annuities and innovative instruments such as tontines—for which the payouts depend on realized survival probabilities—are available. We model individual preferences using a generalized life-cycle utility framework that explicitly incorporates temporal risk aversion, reflecting aversion to uncertainty over the timing of death rather than merely consumption risk. Within this framework, we derive conditions under which individuals with temporal risk aversion prefer a retirement income strategy that combines partial tontinization with partial annuitization, rather than full annuitization.

1. INTRODUCTION

In recent years, risk-sharing schemes—innovative retirement products that allow mortality risks to be shared among providers and/or individuals (cf. OECD 2020, 2022)—have attracted growing attention from both practitioners and academics. In the economics and finance literature, retirement planning has traditionally focused on the demand for annuities across various settings (e.g., Yaari 1965; Davidoff, Brown, and Diamond 2005; Horneff, Maurer, and Stamos 2008; Chai et al. 2011; Bommier, Leroux, and Lozachmeur 2011; Peijnenburg, Nijman, and Werker 2016; Boyer, Box-Couillard, and Michaud 2020). More recent work has begun to explore retirement products that tie benefits to actual mortality outcomes (cf. Bommier and Schernberg 2021). In parallel, the actuarial literature (e.g., Piggott, Valdez, and Detzel 2005; Sabin 2010; Richter and Weber 2011; Milevsky and Salisbury 2015; Chen, Hieber, and Klein 2019) has proposed innovative retirement instruments for which the payouts are directly linked to realized survival probabilities.

A key example of such products is the tontine—a form of risk-sharing arrangement in which a participant's benefits increase as other participants pass away. Originating in the 17th century as public financing tools (Weir 1989), tontines have seen a resurgence in interest due to their ability to pool longevity risk efficiently. Modern tontines, as discussed by Piggott, Valdez, and Detzel (2005) and Sabin (2010), combine features of group annuities with mortality-linked redistributions, offering flexible and cost-effective retirement income solutions. Table 1 presents a selection of tontine implementations in current practice. Products like GuardPath's Modern Tontine are available to individual investors, while Nuovallo's LifePools platform targets retirement providers. These schemes enable institutions to incorporate tontine-style features into their offerings.

In contrast to traditional annuities, where payouts are fixed and insurance providers bear longevity risk, tontines base benefits on realized survival. Providers serve primarily as administrators, without assuming financial liability. As highlighted in OECD (2020), explicit risk-sharing elements such as those in tontines can lead to higher retirement income. A market offering both annuities and mortality-linked products thus allows individuals to benefit from diversification between guaranteed income and longevity-contingent rewards.

We aim to unify the economic and actuarial perspectives by examining retirement decisions under a generalized life-cycle utility framework that accounts for *temporal risk aversion*—a form of risk aversion not only over consumption levels, but also over

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TABLE 1
Overview of Some Tontines Provided in Practice

Name	Provider	Countries	URL
LifePools GuardPath	Nuovallo Ltd.	United States	https://www.nuovallo.com
Longevity Solutions	Guardian Capital	Canada	https://www.guardiancapital.com
Tontine Pension	Tontine Trust	Ireland	https://tontine.com
La Tontine	Le Conservateur	France	https://www.conservateur.fr

the timing of consumption due to uncertain lifespan. This behavioral trait has been supported by empirical evidence (e.g., Delprat, Leroux, and Michaud 2016; Andersen et al. 2018) and plays a critical role in long-term decisions involving lifetime uncertainty.

The classical model of retirement, as introduced by Yaari (1965), assumes that individuals are indifferent to the timing of uncertain lifetime consumption, that is, temporally risk-neutral. Under this assumption, the optimal annuity payout does not depend on the individual's mortality profile. However, as shown by Bommier (2006) and Bommier, Leroux, and Lozachmeur (2011), introducing temporal risk aversion into the utility function alters these results significantly. Individuals prefer smoother consumption paths that account for lifespan uncertainty, which in turn affects the optimal allocation between retirement products.

Building on this framework, we analyze optimal retirement strategies when individuals can access both annuities and tontines. While the literature has previously studied optimal annuitization under temporal risk aversion, we extend these analyses to include mortality-linked products. Using a utility approximation method inspired by Bommier (2006), we derive the optimal demand for tontines. For markets offering only tontines, explicit solutions can be obtained; when both tontines and annuities are present, numerical optimization is required. We also identify bounds for annuity risk loadings that define when mixed investment becomes optimal.

Our findings show that individuals with even moderate temporal risk aversion optimally allocate a share of their wealth to tontines, even in fair annuity markets. In contrast, temporal risk-neutral individuals invest exclusively in annuities (cf. Milevsky and Salisbury 2015). However, full tontinization is never optimal, emphasizing the complementary nature of these products.

Our work is closely related to Bommier, Leroux, and Lozachmeur (2011), Bommier and Schernberg (2021), Chen, Rach, and Sehner (2020), and Weinert and Gründl (2020). We extend Bommier, Leroux, and Lozachmeur (2011) by including tontines in the analysis under temporal risk aversion. While Bommier and Schernberg (2021) use recursive utility to study preferences for adaptive payouts, we focus on the optimal mix of annuities and tontines within a generalized life-cycle model. Unlike Chen, Rach, and Sehner (2020), who assume temporal risk neutrality, and Weinert and Gründl (2020), who apply cumulative prospect theory, our framework emphasizes lifetime uncertainty and its impact on optimal retirement planning.

The article is organized as follows: Section 2 introduces the model and key assumptions. Section 3 presents solutions for optimal retirement planning across different market settings. Section 4 concludes. The appendix provides technical derivations, parameter calibrations, and numerical illustrations.

2. MODEL SETUP

This section introduces the general setting, the underlying assumptions about the mortality, and the lifetime utility of the optimizing economic agent. Following Yaari (1965), we consider a continuous-time model for the retirement decision, in which individuals are only uncertain about their residual lifetime. In particular, throughout the article, we disregard financial market risk to keep the article focused on mortality risk. Let T_x be the random remaining lifetime of an agent aged x years and $\{\mu_{x+t}\}_{t \geq 0}$ be the stochastic force of mortality. Stochastic mortality models are essential for capturing systematic mortality risk, as such risks extend beyond what can be reliably inferred from historical life tables alone. We can then rely on the force of mortality to compute the survival probabilities. To be precise, there is a negative one-to-one relation between mortality rates and survival probabilities. We define conditional and unconditional t -year survival probabilities of an x -year old individual, denoted by $S_x(t)$ and $s_x(t)$, as follows:

$$S_x(t) := \mathbb{P}\left(T_x > t \mid \{\mu_{x+t}\}_{t \geq 0}\right) = e^{-\int_0^t \mu_{x+s} ds}, t \geq 0,$$

$$s_x(t) := \mathbb{E}\left[e^{-\int_0^t \mu_{x+s} ds}\right] = \mathbb{E}[S_x(t)], t \geq 0.$$

The conditional survival probability $S_x(t)$ is observable to agents and retirement product providers at time t , but unknown at the contract-issuing time 0 (except $t = 0$). However, the distributions of T_x and $S_x(t)$ are known by agents and insurers at time 0, which implies that the unconditional survival probability $s_x(t)$ is already known at time 0 as well. We make the following assumption with regard to the force of mortality.

Assumption 2.1. *We assume that $\{\mu_{x+t}\}_{t \geq 0}$ is a nonnegative stochastic process and that $\int_0^\infty \mu_{x+t}(\omega) dt = \infty$. Furthermore, we assume that the process $\{\mu_{x+t}\}_{t \geq 0}$ is sufficiently well behaved that the derivative $S'_x(t)$ of $S_x(t)$ exists.*

This assumption implies that $S_x(t)$ decreases in time t in each path¹ and there is no scenario in which individuals can live forever, that is, $S_x(\infty) = 0$. Note that [Assumption 2.1](#) implies that $s_x(t)$ is also decreasing in t and $s_x(\infty) = 0$ as well.

In the following, let us specify the risk preferences of the optimizing agent. In contrast to the majority of the literature in this field (Yaari (1965) and many follow-up studies like, for example, Milevsky and Salisbury (2015) and Peijnenburg, Nijman, and Werker (2016), our preference modeling is inspired by Bommier (2006) and Bommier, Leroux, and Lozachmeur (2011), in which individuals are not only risk averse about consumption, but also (possibly) risk averse about their lifetime (so-called temporal risk aversion). As the retirement plans considered in this study only contain mortality risk, allowing for different types of risk aversion about the lifetime makes it possible to conduct a comprehensive analysis about the demand for retirement products. Let $\alpha(t)$ be a subjective discount factor, u be a strictly increasing and concave utility function, and $C = \{C(t)\}_{t \geq 0}$ be a (possibly stochastic) single-dimensional consumption profile. The lifetime utility of an agent with a remaining lifetime of T_x years is then given by

$$U(C, T_x) = \Phi \left(\int_0^{T_x} \alpha(t) u(C(t)) dt \right). \quad (1)$$

In the typical expected lifetime utility framework (e.g., Yaari 1965), agents have a linear lifetime utility, modeled through a linear Φ . At each moment in life, they derive instantaneous utility $\alpha(t)u(C(t))$, which is then additively accumulated from time 0 until the moment of death T_x to determine their total lifetime utility. However, since lifetime is a random variable that follows a known distribution, agents cannot determine their lifetime utility ex ante, as it depends on both the stochastic nature of life duration and future consumption. By employing a general function of Φ , we are able to incorporate risk aversion with respect to the randomness in the future lifetime.² In the typical expected lifetime utility framework where a linear Φ is used, the agent is risk neutral about the lifetime, while a concave function Φ would indicate a positive risk aversion about his/her lifetime.

Assumption 2.2. *We assume that Φ is twice differentiable satisfying $\Phi' > 0$, $\Phi'' \leq 0$ and $\Phi(0) \in (-\infty, \infty)$.*

As a direct consequence of [Assumption 2.2](#), we assume that $\Phi(0) = 0$ without loss of generality.³

We consider both temporal risk-averse and temporal risk-neutral individuals, as the latter can form a comparison basis and help us examine the impact of temporal risk aversion for retirement decisions. There are some recent studies examining whether individuals exhibit temporal risk aversion or temporal risk neutrality. For instance, Cheung (2015), Delprat, Leroux, and Michaud (2016), and Andersen et al. (2018) confirm that some individuals exhibit temporal risk aversion rather than temporal risk neutrality.

Taking the expectation of [Equation \(1\)](#), we obtain the expected lifetime utility function as

$$EU(C) = \mathbb{E}[U(C, T_x)] = \mathbb{E} \left[\Phi \left(\int_0^{T_x} \alpha(t) u(C(t)) dt \right) \right]. \quad (2)$$

Andersen et al. (2018) consider an expected lifetime utility similar to [Equation \(2\)](#) in discrete time. DeJarnette et al. (2020) call the preference in [Equation \(2\)](#) a generalized expected discounted utility model and show that it results from combining the axioms of expected utility with the axioms of discounted utility. Note that the utility preferences in [Equation \(2\)](#) particularly differ from those in, for example, Kreps and Porteus (1978), Epstein and Zin (1989), Weil (1990), and Bommier and

¹For example, the probability of surviving the next 20 years should always be smaller than the probability of surviving the next 10 years.

²It is important to note that our framework differs from the smooth ambiguity approach à la Klbanoff, Marinacci, and Mukerji (2012), as we do not assume that the policyholder forms multiple subjective distributions over survival probabilities or exhibits ambiguity aversion toward such uncertainty.

³Assume that $\Phi(0) \in (-\infty, \infty) \setminus \{0\}$. Then, instead of considering $\Phi(c)$, we can consider $\tilde{\Phi}(c) := \Phi(c) - \Phi(0)$, which resembles preferences identical to Φ and fulfills $\tilde{\Phi}(0) = 0$.

Schernberg (2021), who consider recursive utility preferences at *each* time, allowing individuals to dynamically adjust their consumption profile over time. We, on the other hand, consider a life-cycle model as in, for example, Yaari (1965), Bommier (2006), and Bommier, Leroux, and Lozachmeur (2011), and only consider the expected utility at time 0. In particular, all decisions made by agents are made at time 0 and, although information on mortality prospects becomes available, we assume that agents cannot take any actions after time 0. For further details on this topic, we refer interested readers to Cohen et al. (2020), who provide a literature overview on measuring temporal preferences.

Finally, in this article, we limit ourselves to utility preferences regarding consumption of the constant relative risk aversion (CRRA) class, following previous life-cycle studies (e.g., Cocco and Gomes 2012; Hubener, Maurer, and Mitchell 2016; Peijnenburg, Nijman, and Werker 2016; Kim, Maurer, and Mitchell 2016). According to Sharpe (2017), CRRA utility is the most frequently used utility function to capture the preferences of individuals.

Assumption 2.3. *We assume that $\frac{u''(c)c}{u'(c)}$ is independent of c , that is, we assume a constant intertemporal elasticity of substitution. This is equivalent to assuming a constant relative risk aversion, resulting in*

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma}, \gamma \neq 1, \gamma > 0, \\ \ln(c), \gamma = 1, \end{cases}$$

where γ is the constant degree of relative risk aversion.

3. INDIVIDUALS' OPTIMAL CONSUMPTION

In this section, we consider the optimal consumption problem of an agent in a market that exclusively deals with tontine products, a market that exclusively deals with annuity products, and a market containing both products. In particular, we introduce the following simplifying assumption.

Assumption 3.1. *We assume that individuals do not hold any other assets—such as risk-free securities or equities—outside the retirement plans under consideration.*

In the following subsection, we start with the consideration of tontines. Note that optimal consumption problems in markets exclusively dealing with tontines have been analyzed, for example, in Milevsky and Salisbury (2015), Bernhardt and Donnelly (2019), Chen, Hieber, and Klein (2019), and Chen, Rach, and Sehner (2020). However, these studies do not account for temporal risk aversion. Under a linear lifetime utility, these articles analyze the impact of risk loadings, subjective mortality beliefs, and bequest motives on the optimal consumption.

3.1. Tontines

To analyze how adding a risk-sharing component to retirement plans can potentially increase individuals' welfare where the individuals exhibit temporal risk aversion, we take a tontine as an example. In a tontine contract, a homogeneous pool made up of a large number of agents, who are assumed to be identical copies of each other, shares the mortality risks. In such a pool, the idiosyncratic risk can be diversified by choosing the pool size properly large (cf. Qiao and Sherris 2013; Bernhardt and Donnelly 2021). The systematic mortality risk, however, affects all the agents in the same direction and can therefore not be diversified.

To maintain the tractability of our model, we introduce [Assumption 3.2](#).

Assumption 3.2. *We assume an infinitely large tontine pool size.*

Note that this assumption implies that agents' mortality follows the conditional survival curve $S_x(t)$ precisely. Building upon the finite-pool homogeneous tontine introduced by Milevsky and Salisbury (2015) and applying the conditional law of large numbers (cf. Majerek, Nowak, and Zieba 2005) under the assumption of conditional independence of future lifetimes, we obtain a consumption level of $C(t) = d(t)/S_x(t)$ for a single individual, where $d(t)$ is a deterministic payout function specified at time 0.⁴ As the conditional survival probability $S_x(t)$ is a random process, the future consumption stream is random, compared to regular annuity products with deterministic consumption profiles.

⁴Note that the number of living participants in a finite pool whose members are conditionally independent of each other follows a conditional binomial distribution $(N(t)|\{\mu_{x+t}\}_{t \geq 0}) \sim \text{Bin}(n, S_x(t))$. As a consequence, conditional on $\{\mu_{x+t}\}_{t \geq 0}$, a tontine delivering the payoff $nd(t)/N(t)$ to a living policyholder tends to $d(t)/S_x(t)$ as $n \rightarrow \infty$, as $(N(t)/n|\{\mu_{x+t}\}_{t \geq 0}) \rightarrow S_x(t)$.

Now consider an agent endowed with an initial wealth W_0 that will be invested in such a tontine. To determine the single up-front premium, we assume that the insurer relies on the expected value principle with a proportional loading $\delta_d \geq 0$. This means that the insurer charges the expected present value of future benefits along with a proportional risk loading. The premium charged for the tontine is therefore given by

$$P_0^d = (1 + \delta_d) \mathbb{E} \left[\int_0^\infty B(t) \cdot S_x(t) \cdot \frac{d(t)}{S_x(t)} dt \right] = (1 + \delta_d) \int_0^\infty B(t) \cdot d(t) dt,$$

where we assume that $B(t)$ is a deterministic discount function from time t to time 0. Taking account of the budget constraint, that is, $P_0^d \leq W_0$, we arrive at the following optimization problem for the agent at time 0:

$$\begin{aligned} \max_{d(t)} EU \left(\left\{ \frac{d(t)}{S_x(t)} \right\}_{t \geq 0} \right) &= \mathbb{E} \left[\Phi \left(\int_0^{T_x} \alpha(t) u \left(\frac{d(t)}{S_x(t)} \right) dt \right) \right] \\ \text{subject to } P_0^d &\leq W_0. \end{aligned} \quad (3)$$

For risk preferences with temporal risk aversion described through the general function Φ , this optimization problem cannot be solved analytically, in contrast to the case where the agent is risk neutral about his/her lifetime (with a linear Φ). In the following, we approximate the expected lifetime utility using a method similar to that of Bommier, Leroux, and Lozachmeur (2011) for approximating annuities. We extend the analysis of Bommier, Leroux, and Lozachmeur (2011) nontrivially to tontines and systematic mortality risk.

Throughout the article, we rely on the following additive approximation of the expected lifetime utility:

$$EU \left(\left\{ \frac{d(t)}{S_x(t)} \right\}_{t \geq 0} \right) \approx \mathbb{E} \left[\int_0^\infty S_x(t)^\gamma \beta(t) \alpha(t) u(d(t)) dt \right], \quad (4)$$

where

$$\beta(t) := -\frac{1}{S_x(t)} \int_t^\infty S'_x(s) \cdot \Phi' \left(\int_0^s \alpha(\tau) d\tau \right) ds. \quad (5)$$

The key assumption underlying this approximation is the notion of a “priceless life context” (Bommier 2006; Bommier, Leroux, and Lozachmeur 2011), which posits that the welfare difference between being alive and being deceased far exceeds the utility difference between varying levels of consumption. The technical derivation of this approximation is provided in [Appendix A.1](#) and a numerical analysis of the accuracy of this approximation is provided in [Appendix C](#).

[Equation \(4\)](#) states that we can transfer the expected lifetime utility of an agent exhibiting temporal risk aversion and owning a subjective discount factor $\alpha(t)$ to the expected lifetime utility function of an agent exhibiting temporal risk neutrality, but now owning a different subjective discount factor $\beta(t)\alpha(t)$. Note the following remarks about the term $\beta(t)$:

- If Φ is a linear function, $\beta(t)$ is equal to a constant. This corresponds to the case in which the agent has a linear lifetime utility, demonstrating temporal risk neutrality. The resulting preference is the one used, for example, in Yaari’s pioneering article (Yaari 1965).
- If Φ is a concave function, that is, if individuals exhibit temporal risk aversion, we find that $\beta(t)$ decreases in t .⁵ This means that temporal risk aversion makes individuals behave more impatiently and seek consumption at early rather than late retirement ages.

We now consider the individual optimization problem assuming that the individual has exclusive access to tontines. Using the approximation (4), we arrive at the following optimization problem:

⁵This can be seen by noting that $\Phi'' < 0$ and [Assumption 2.1](#) imply $\beta'(t) = -\frac{S'_x(t)}{S_x(t)} \int_t^\infty \frac{S(s)}{S_x(s)} \Phi'' \left(\int_0^s \alpha(\tau) d\tau \right) ds < 0$.

$$\begin{aligned} & \max_{d(t)} \int_0^\infty \mathbb{E}[S_x(t)^\gamma \beta(t)] \alpha(t) u(d(t)) dt \\ & \text{subject to } P_0^d \leq W_0. \end{aligned} \quad (6)$$

Theorem 3.3 provides the solution to problem (6).

Theorem 3.3. *The solution to the optimization problem (6) is*

$$d^*(t) = \left(\frac{(1 + \delta_d) \lambda_d B(t)}{\mathbb{E}[S_x(t)^\gamma \beta(t)] \alpha(t)} \right)^{-\frac{1}{\gamma}}, \quad (7)$$

where the Lagrangian multiplier λ_d is, for all $\gamma > 0$, given by

$$\lambda_d = \left(\frac{1 + \delta_d}{W_0} \int_0^\infty B(t) \cdot \left(\frac{(1 + \delta_d) B(t)}{\mathbb{E}[S_x(t)^\gamma \beta(t)] \alpha(t)} \right)^{-\frac{1}{\gamma}} dt \right)^\gamma.$$

For $\gamma \neq 1$, the expected discounted lifetime utility is then given by

$$U_d = \frac{\lambda_d}{1 - \gamma} \cdot W_0.$$

Proof: See Appendix A.2.

Intuitively, the optimal tontine payout function should be decreasing over time to offset the (potentially) rapid increase in retirement benefits generated by the mortality risk-sharing component $1/S_x(t)$. Only if individuals are extremely patient and require much higher consumption levels at older ages than at younger ages could the payout function be constant or even increasing. These patterns can be observed in the optimal payout function:

- The optimal payout function under temporal risk neutrality (linear Φ) is given by

$$d^*(t) = \left(\frac{(1 + \delta_d) \lambda_d B(t)}{\mathbb{E}[S_x(t)^\gamma] \alpha(t)} \right)^{-\frac{1}{\gamma}},$$

where

$$\lambda_d = \left(\frac{1 + \delta_d}{W_0} \int_0^\infty B(t) \cdot \left(\frac{(1 + \delta_d) B(t)}{\mathbb{E}[S_x(t)^\gamma] \alpha(t)} \right)^{-\frac{1}{\gamma}} dt \right)^\gamma.$$

Further assuming $\alpha(t)/B(t)$ to be nonincreasing, the payout function therefore decreases in t , as has already been observed in the literature (e.g. Milevsky and Salisbury 2015). The term $\alpha(t)/B(t)$ is nonincreasing if individuals' subjective discount factors $\alpha(t)$ lead to a stronger or equal subjective discounting does than the discount factor used by the insurer (given by $B(t)$). However, if we assume that individuals are more patient than they would be under the discount factor $B(t)$, it is not generally clear how the function $\alpha(t)\mathbb{E}[S_x(t)^\gamma]/B(t)$ will evolve over time, and the optimal payout function may therefore be constant or increasing as well.

- If $\beta(t)$ is decreasing in t , $\mathbb{E}[S_x(t)^\gamma \beta(t)]$ decreases more steeply than $\mathbb{E}[S_x(t)^\gamma]$, that is., temporal risk aversion leads to a more steeply decreasing payout function than temporal risk neutrality. This result is economically plausible because temporal risk aversion leads to time discounting in the presence of mortality risk (see also Bommier 2006).

To gain further insights on the economic implications of temporal risk aversion in tontines, we consider a numerical example. Throughout the numerical studies of this article, we rely on a shocked Gompertz model. For details on the

(deterministic) Gompertz model, see Gompertz (1825) or, for example, Gumbel (1958). Following, for instance, Lin and Cox (2005), we apply a random shock ε to the Gompertz mortality rates, assuming that ε is a continuous random variable with density $f_\varepsilon(\cdot)$ and support on $(-\infty, 1)$. Mortality rates are then given by

$$\mu_{x+t} = (1 - \varepsilon) \frac{1}{b} e^{\frac{x+t-m}{b}}, \quad (8)$$

where b is the dispersion coefficient and m is the modal age at death. This model is motivated by the fact that, in many countries, insurers are required by regulation to test their balance sheets against a variety of shock scenarios. This is, for example, the case in the United States, Canada, and the European Union. Additionally, this model has a very simple interpretation: Positive shocks ($1 > \varepsilon > 0$) increase life expectancy, which could, for example, be caused by a medical breakthrough. Negative shocks ($\varepsilon < 0$), on the other hand, decrease life expectancy, as potentially caused, for example, by a pandemic. For $\varepsilon \equiv 0$, we obtain the (deterministic) Gompertz model as a special case. Furthermore, we rely on the parameters summarized in Table 2, which are calibrated in such a way that a reasonable set of benchmark parameters is obtained. Appendix B provides details on the calibration of our model.

Figure 1 shows the behavior of $\beta(t)$ depending on the mortality shock ε and the time t .

As already stated, $\beta(t)$ decreases in t for all values of ε . Additionally, we observe that $\beta(t)$ decreases in the mortality shock ε . More importantly, we observe that $\beta(t)$ decreases more steeply in t for smaller values of ε . This means that individuals behave more impatiently under smaller and/or negative mortality shocks, that is, when life expectancy is low, and preferences for consumption at younger ages are stronger. Under larger and/or positive mortality shocks, that is, when life expectancy is high, however, individuals behave more patiently and seek more consumption at older ages. In other words, with a tontine, individuals exhibiting temporal risk aversion seek a diversification strategy that allows for consumption to be reduced if survival probabilities are high and consumption to be increased under lower life expectancy (see also Bommier and Schernberg 2021).

TABLE 2
Benchmark Parameters

Initial wealth $W_0 = 100$	Temporal risk aversion $\Phi(y) = \frac{1}{\theta} - \frac{1}{\theta} e^{-\theta y}$, $\theta = 0.035$	Utility function $u(y) = \frac{y^{1-\gamma}}{1-\gamma}$, $\gamma = 3$
Discount function $B(t) = e^{-0.01t}$	Subjective discount rate $\alpha(t) = e^{-0.01t}$	Risk loading $\delta_d = 0$
Initial age $x = 65$	Gompertz law $m = 88.721$, $b = 10$	Mortality shock $\varepsilon \sim \mathcal{N}_{(-\infty, 1)}(-0.0035, 0.0814^2)$

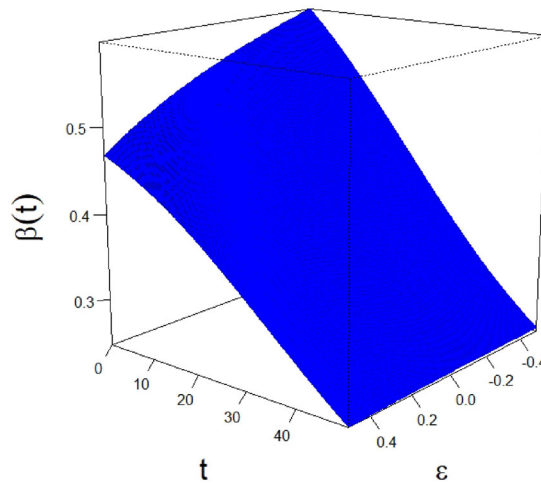


FIGURE 1. $\beta(t)$ Depending on the Time t and the Mortality Shock ε . Note: The parameters are chosen as in Table 2.

In Figure 2, panel (a), we show the optimal payout functions under temporal risk neutrality and temporal risk aversion. By varying the parameter θ , we can study the consumption profiles of different agents who do not have an additive lifetime utility. If θ approaches zero (consequently, Φ' going to 1), it makes the lifetime utility function closer to the additive utility function. During the first 10 years, the payout function under temporal risk aversion is slightly above that obtained under temporal risk neutrality. After 10 years, this relation is reversed. If θ grows larger, the resulting lifetime utility moves further away from the linear lifetime utility, and the optimal consumption deviates more from the optimal payout function under temporal risk neutrality. To summarize, risk aversion about one's lifetime works similarly to a larger subjective discounting rate, in line with the remarks made about $\beta(t)$ following Equation (5).

In Figure 2, panel (b), we show the optimal consumption $d^*(t)/S_x(t)$ in the tontine. Since the consumption in a tontine is random, we show the median as well as the 10% and 90% quantile range. In this panel, we observe that consumption remains rather stable during younger retirement ages and then potentially increases rather drastically at extremely old retirement ages ($t \geq 40$, i.e., age 105 and above). This is due to the fact that extremely small survival probabilities at old retirement ages result, leaving to the few surviving individuals in the pool large retirement benefits. Additionally, we observe that payoffs are (potentially) more volatile at older retirement ages compared to younger retirement ages. Similar to panel (a), we also observe that a higher degree of temporal risk aversion leads individuals to prefer higher consumption at earlier ages.

3.2. Annuities

In this section, we consider a deterministic consumption profile $C(t) = c(t)$. We revisit and expand upon findings from the study by Bommier, Leroux, and Lozachmeur (2011) with some modifications: We employ broader stochastic mortality rates instead of deterministic ones. In addition, we introduce a novel extension by incorporating generalized risk loadings into all aspects of our analysis. When we eliminate systematic mortality risk and set the risk loadings to zero, our framework reverts to the scenario described in Bommier, Leroux, and Lozachmeur (2011). Optimal annuitization has been frequently studied in the literature, for example, in Yaari (1965), Davidoff, Brown, and Diamond (2005), and Peijnenburg, Nijman, and Werker (2016). The articles closest to ours are Bommier (2006) and Bommier, Leroux, and Lozachmeur (2011), as they consider temporal risk aversion.

Denoting by δ_a the proportional loading charged for an annuity, the premium charged for an annuity is given by

$$P_0^a = (1 + \delta_a) \int_0^\infty B(t) \cdot s_x(t) \cdot c(t) dt.$$

Assuming that an agent endowed with the initial wealth W_0 invests exclusively in annuities, we arrive at the following optimization problem:

$$\begin{aligned} \max_{c(t)} EU(\{c(t)\}_{t \geq 0}) &= \mathbb{E} \left[\Phi \left(\int_0^{T_x} \alpha(t) u(c(t)) dt \right) \right] \\ \text{subject to } P_0^a &\leq W_0. \end{aligned}$$

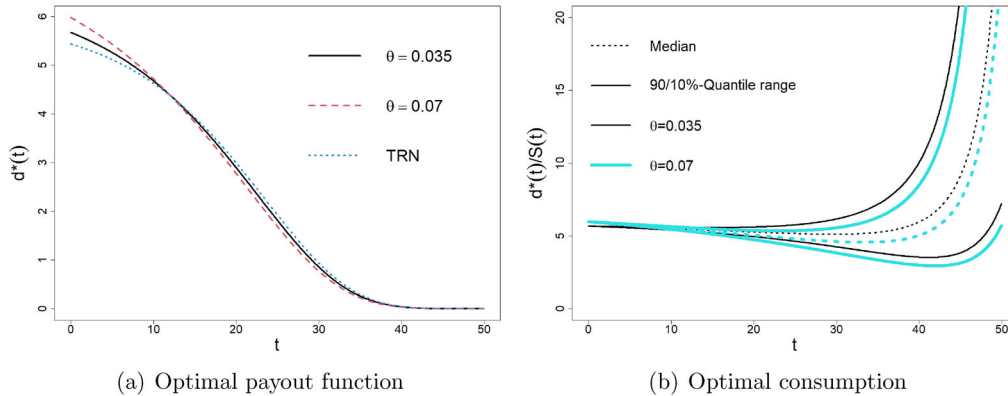


FIGURE 2. Optimal Payout Function $d^*(t)$ and Consumption $d^*(t)/S_x(t)$ over Time under Temporal Risk Aversion and Temporal Risk Neutrality (TRN). Note: The parameters are chosen as in Table 2.

Similar to the tontine case, we approximate the expected lifetime utility function with temporal risk aversion by a temporal risk-neutral preference. For this, we define the term

$$\bar{\beta}(t) := -\frac{1}{s_x(t)} \int_t^\infty s'(s) \cdot \Phi' \left(\int_0^s \alpha(\tau) d\tau \right) ds, \quad (9)$$

which is related to $\beta(t)$ by [Lemma A.1](#). Following analogous steps as in [Appendix A.1](#), and applying [Lemma A.1](#), we end up with the following additive approximation of the expected lifetime utility function:

$$EU \left(\{c(t)\}_{t \geq 0} \right) \approx \int_0^\infty s_x(t) \bar{\beta}(t) \alpha(t) u(c(t)) dt. \quad (10)$$

We can again transfer the expected lifetime utility of an agent showing temporal risk aversion and owning a subjective discount factor $\alpha(t)$ to the expected lifetime utility of an agent showing temporal risk neutrality, but now owning a different subjective discount factor $\bar{\beta}(t)\alpha(t)$. In this sense, conclusions can be drawn here analogous to those for the tontine. In particular, note the following remarks about $\bar{\beta}(t)$:

- If Φ is a linear function, $\bar{\beta}(t)$ is equal to a constant. This particular case deals with temporal risk neutrality.
- If Φ is concave, that is, if individuals exhibit temporal risk aversion, $\bar{\beta}(t)$ decreases in t .⁶ Thus, temporal risk aversion again leads individuals to behave less patiently, implying higher consumption at earlier retirement ages.

Given the approximation (10), we now arrive at the following optimization problem:

$$\begin{aligned} & \max_{c(t)} \int_0^\infty s_x(t) \bar{\beta}(t) \alpha(t) u(c(t)) dt \\ & \text{subject to } P_0^a \leq W_0. \end{aligned} \quad (11)$$

We are again able to solve this optimization problem explicitly.

Theorem 3.4. *The solution to optimization problem (11) is*

$$c^*(t) = \left((1 + \delta_a) \frac{\lambda_a B(t)}{\bar{\beta}(t) \alpha(t)} \right)^{-\frac{1}{\gamma}}, \quad (12)$$

where the Lagrangian multiplier λ_a is, for all $\gamma > 0$, given by

$$\lambda_a = \left(\frac{1 + \delta_a}{W_0} \int_0^\infty B(t) \cdot s_x(t) \cdot \left(\frac{(1 + \delta_a) B(t)}{\bar{\beta}(t) \alpha(t)} \right)^{-\frac{1}{\gamma}} dt \right)^\gamma.$$

For $\gamma \neq 1$, the expected discounted lifetime utility is then given by

$$U_a = \frac{\lambda_a}{1 - \gamma} \cdot W_0.$$

Proof: See [Appendix A.4](#).

There are the following special cases:

- In the special case where $\alpha(t) = B(t)$ and Φ is a linear function, the optimal payout function $c^*(t)$ is constant. Setting the risk loading of the annuity equal to zero, we obtain a well-known result that has already been stated in Yaari (1965):

⁶This can be seen by noting that $\bar{\beta}'(t) = -\frac{s'(t)}{s_x(t)} \int_t^\infty \frac{s(s)}{s_x(s)} \Phi'' \left(\int_0^s \alpha(\tau) d\tau \right) ds$, which is smaller than zero if we assume $\Phi'' < 0$.

Constant annuitization is optimal if there is no bequest motive. Additionally, if $\alpha(t)/B(t)$ decreases (i.e., if agents are more impatient than they would be under the subjective discount factor), the optimal consumption decreases. However, if $\alpha(t)/B(t)$ is increasing (i.e., if agents are more patient than they would be under the subjective discount factor), the optimal consumption increases over time.

- Similar to the tontine, being risk averse about their lifetime makes agents less patient: They prefer higher consumption at earlier retirement ages. In particular, if $\alpha(t)/B(t)$ is nonincreasing and Φ is concave, the optimal consumption profile decreases over time, because $\beta(t)$ decreases in t . If, however, $\alpha(t)/B(t)$ is increasing, the time-discounting preferences of the agent may interfere with the temporal risk aversion and lead to a weaker subjective discounting.

In Figure 3, we show the optimal consumption profiles for $\delta_a = 0$, that is, assuming fair pricing. This assumption does not reflect reality properly, but is frequently made when analyzing the optimal consumption profile throughout retirement (cf. Milevsky and Salisbury 2015; Bommier, Leroux, and Lozachmeur 2011; Bommier and Schernberg 2021).

We observe that, under temporal risk aversion, consumption decreases over time. Note that if Φ is a linear function, the optimal consumption is constant and equal to $W_0 / \int_0^\infty e^{-rt} s_x(t) dt = 5.45$ (assuming the benchmark parameters in Table 2). Hence, we can draw observations similar to those from Figure 2, namely, that the presence of temporal risk aversion leads agents to behave less patiently. In summary, individuals with temporal risk aversion may not perceive annuity products with constant benefits, which are frequently traded, as optimal choices. This finding could provide insight into the limited participation of individuals in annuity markets. In the forthcoming analysis, we will investigate whether the lifetime utility of temporal risk-averse individuals can be enhanced by introducing the option to trade in tontines.

3.3. Combined Investment

In contrast to the existing literature in this field (e.g. Yaari 1965; Davidoff, Brown, and Diamond 2005; Horneff, Maurer, and Stamos 2008; Chai et al. 2011; Bommier, Leroux, and Lozachmeur 2011; Peijnenburg, Nijman, and Werker 2016), we are interested in finding out whether annuities can be supplemented by a mortality-linked product for an individual showing temporal risk aversion. In other words, we assume that agents have access to both tontines and annuities and aim to find out whether and how these individuals can benefit from a combined investment in both plans. In this respect, our article is particularly related to Chen, Rach, and Sehner (2020) and Weinert and Gründl (2020), which also determine the optimal portfolio consisting of tontines and annuities. However, Chen, Rach, and Sehner (2020) adopt a linear lifetime utility and Weinert and Gründl (2020) use multi cumulative prospect theory.

A portfolio consisting of an annuity and a tontine results in the following consumption level of a living agent at each time $t \geq 0$:

$$C(t) = c_{ad}(t) + \frac{d_{ad}(t)}{S_x(t)}.$$

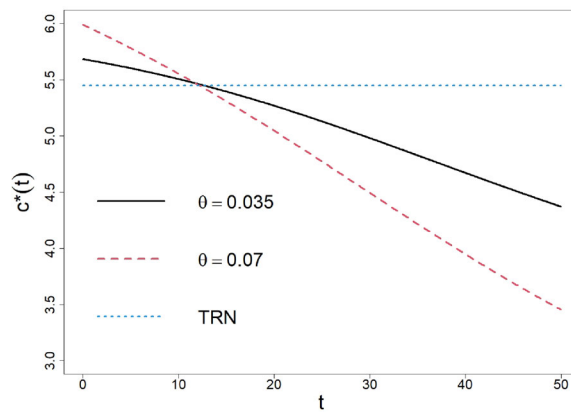


FIGURE 3. Optimal Consumption Profile $c^*(t)$ over Time under Temporal Risk Aversion and Temporal Risk Neutrality (TRN). *Note:* The parameters are chosen as in Table 2 and we set $\delta_a = 0$.

The premium for such a portfolio consisting of $\{c_{ad}(t)\}_{t \geq 0}$ and $\{d_{ad}(t)/S_x(t)\}_{t \geq 0}$ is given by $P_0^{a,ad} + P_0^{d,ad}$, where

$$\begin{aligned} P_0^{a,ad} &= (1 + \delta_a) \int_0^\infty B(t) s_x(t) c_{ad}(t) dt, \\ P_0^{d,ad} &= (1 + \delta_d) \int_0^\infty B(t) \cdot d_{ad}(t) dt \end{aligned}$$

are the premiums of the annuity and the tontine in the portfolio, respectively. Next, we present the optimization problem for an agent endowed with W_0 who aims to invest in such a portfolio:

$$\begin{aligned} \max_{c_{ad}(t), d_{ad}(t)} \quad & \mathbb{E} \left[\Phi \left(\int_0^{T_x} \alpha(t) u \left(c_{ad}(t) + \frac{d_{ad}(t)}{S_x(t)} \right) dt \right) \right] \\ \text{subject to} \quad & P_0^{a,ad} + P_0^{d,ad} \leq W_0. \end{aligned}$$

In this optimization problem, agents maximize the expected discounted lifetime utility simultaneously over all possible annuity and tontine payout functions. In particular, agents determine not only the optimal structure of the corresponding payout functions, but also the optimal fractions of wealth invested in the annuity and the tontine. To solve this optimization problem, we rely on the same approximation technique as in the previous subsections. Following the steps in [Appendix A.1](#), we arrive at the following approximate optimization problem:

$$\begin{aligned} \max_{c_{ad}(t), d_{ad}(t)} \quad & \mathbb{E} \left[\int_0^\infty S_x(t) \beta(t) \alpha(t) u \left(c_{ad}(t) + \frac{d_{ad}(t)}{S_x(t)} \right) dt \right] \\ \text{subject to} \quad & P_0^{a,ad} + P_0^{d,ad} \leq W_0. \end{aligned} \tag{13}$$

The expected lifetime utility from the combined investment can again be approximated through a linear lifetime utility, as in [Equation \(13\)](#). However, the optimal solution to the annuity and tontine payoff can still not be determined analytically, as the utility function of a sum is unequal to the sums of the separate utility levels for power utility. Hence, we are unable to derive an explicit optimal consumption profile for optimization problem (13). Nevertheless, depending on the magnitude of the risk loadings, we can still derive theoretical results on the comparison of annuities, tontines, and a portfolio of the two, and learn when it is optimal to invest in a nontrivial portfolio, that is, a positive amount in both tontines and annuities. In [Theorem 3.5](#), we summarize the results.

Theorem 3.5.

1. The optimal solution to problem (13) is given by $c_{ad}(t) = c^*(t)$, $d_{ad}(t) = 0$; that is, it is optimal to invest all the initial wealth in the optimal annuity if and only if

$$\delta_a \leq \underline{\delta}_a := \inf_{t \geq 0} \frac{\bar{\beta}(t)}{\mathbb{E}[\beta(t)]} (1 + \delta_d) - 1, \tag{14}$$

where $\beta(t)$ and $\bar{\beta}(t)$ are defined in [Equations \(5\) and \(9\)](#), respectively.

2. The optimal solution to problem (13) is given by $d_{ad}(t) = d^*(t)$, $c_{ad}(t) = 0$; that is, it is optimal to invest all the initial wealth in the optimal tontine if and only if

$$\delta_a \geq \bar{\delta}_a := \sup_{t \geq 0} \frac{\mathbb{E}[S_x(t)^{\gamma+1} \beta(t)]}{\mathbb{E}[S_x(t)^\gamma \beta(t)] S_x(t)} (1 + \delta_d) - 1. \tag{15}$$

3. If both conditions (14) and (15) are not fulfilled, it is optimal for the agent to invest positive amounts in both the annuity and the tontine.

Proof: See [Appendix A.5](#).

Remark 3.6. In the special case where Φ is a linear function, $\mathbb{E}[\beta(t)]$ and $\bar{\beta}(t)$ are equal to the same constant. Therefore, [Equation \(14\)](#) can be rearranged to $\delta_a \leq \delta_d$ and [Equation \(15\)](#) simplifies to

$$\delta_a \geq \sup_{t \geq 0} \frac{\mathbb{E}[S_x(t)^{\gamma+1}]}{\mathbb{E}[S_x(t)^\gamma] S_x(t)}.$$

In particular, Equation (14) is a natural extension to Proposition 3.3 in Chen, Rach, and Sehner (2020), in which finite pool sizes and linear lifetime utility functions are considered. In the presence of temporal risk aversion, the demand for annuities and tontines can both be either increased or decreased, depending on the mortality law and the resulting structure of $\beta(t)$, $\bar{\beta}(t)$, and $\mathbb{E}[S_x(t)^\gamma \beta(t)]$. In particular, if there exists at least one $t > 0$ such that $\bar{\beta}(t) < \mathbb{E}[\beta(t)]$, Equation (14) implies that $\underline{\delta}_a < \delta_a$. In other words, for a temporal risk averse agent, the loading of an annuity shall be unrealistically low (even lower than the loading of a tontine) to make the agent choose full annuitization. In this case, temporal risk aversion therefore increases the attractiveness of tontines and leads them to prefer partial tontinization with partial annuitization to full annuitization.

With tontines, individuals' consumption profiles depend significantly on the systematic mortality risk. With annuities, on the other hand, consumption is deterministic. In the fraction $\bar{\beta}(t)/\mathbb{E}[\beta(t)]$ in Equation (14), individuals' temporal preferences with respect to the tontine and the annuity are compared. Applying the parameters in Table 2, we analyze the critical bounds of the annuity loading of Equations (14) and (15) numerically (see Table 3).

We observe that $\underline{\delta}_a$ is below 0 and that it decreases in the degree of temporal risk aversion θ . The fact that the critical value of $\underline{\delta}_a$ is negative for all values $\theta > 0$ considered in Table 3 means that investing exclusively in annuities is not optimal, as long as a nonnegative risk loading is charged for the annuity. In other words, even in a perfect annuity market, it is not optimal to invest all the initial wealth in the annuity if temporal risk aversion is present. This conclusion can, under the benchmark parameters in Table 2, be drawn for any degree of relative risk aversion with respect to consumption γ , as Equation (14) is independent of γ . Furthermore, $\bar{\delta}_a = \infty$ tells us that no agent is willing to invest exclusively in tontines, regardless of the temporal risk aversion. Based on this numerical analysis, we can again conclude that tontines are more attractive to individuals who are subject to temporal risk aversion and tontines become more attractive as the degree of temporal risk aversion increases.

To illustrate the increased attractiveness of tontines under temporal risk aversion further, Figure 4 provides the optimal payout functions in the portfolio for an agent with temporal risk neutrality and with temporal risk aversion.

The result in panel (a) has already been derived theoretically in Chen, Rach, and Sehner (2020): Under fair insurance, full annuitization and zero tontinization is optimal. In panel (b), however, we observe that during the first few years of retirement, the agent receives only tontine payments and no annuity payments. Then the agent receives a mixture of both, where the payout function in the tontine approaches zero and rather high payments from the annuity are received. Tontinization is particularly

TABLE 3

Analysis of the Bounds (14) and (15) Depending on the Temporal Risk Aversion θ , with the Parameters Chosen as in Table 2

θ	TRN	0.035	0.07	0.105	0.14
$\underline{\delta}_a$	0	-0.0014	-0.0027	-0.0039	-0.0051
$\bar{\delta}_a$	∞	∞	∞	∞	∞

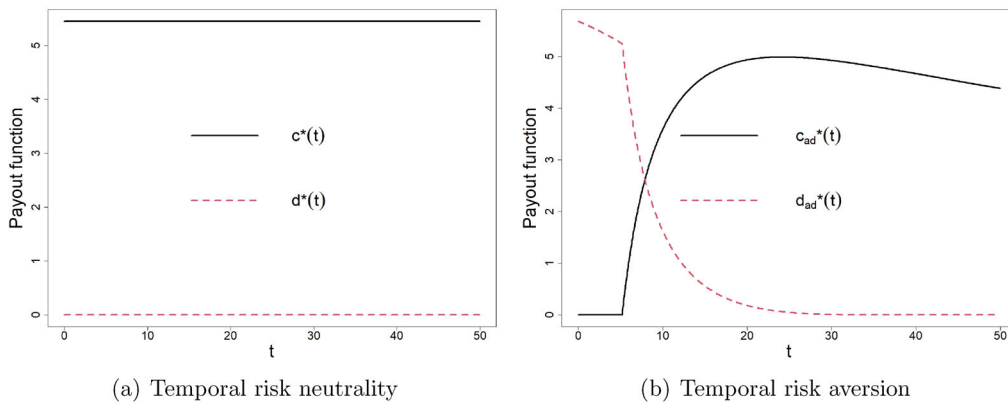


FIGURE 4. Optimal Payout Functions in the Portfolio for an Agent Exhibiting Temporal Risk Aversion. Note: The parameters are chosen as in Table 2 and we set $\delta_a = 0$.

attractive at early retirement ages. Based on the optimal payments $c_{ad}^*(t)$ and $d_{ad}^*(t)$, we are able to compute the present values of the annuity and of the tontine, respectively. Under temporal risk aversion, the overall fraction of wealth invested in the annuity is given by 52%, meaning that the rather moderate level of temporal risk aversion $\theta = 0.035$ along with $\gamma = 3$ already reduces the fraction of wealth optimally invested in an annuity by 48%.

To demonstrate this result in a broader generality, Table 4 shows a sensitivity analysis of the optimal fraction of initial wealth invested in the annuity.

First, we observe that for all combinations of temporal risk aversion $\theta > 0$ and consumption risk aversion $\gamma > 0$, the optimal fraction invested in the annuity lies below 100%. The reason for this is found in Equation (14) and Table 3: As Equation (14) does not depend on γ , we can already conclude from the results in Table 3 that, under the benchmark parameters in Table 2, for any degree of relative risk aversion γ , the optimal fraction of wealth annuitized lies below 100% for any positive value of the absolute degree of temporal risk aversion θ considered in Table 3. In other words, the degree of temporal risk aversion and the risk loadings of the annuity and the tontine determine for any degree of risk aversion with respect to consumption γ whether it is optimal to fully annuitize or not. If full annuitization is not optimal, the degree of relative risk aversion γ does, however, determine the fraction of wealth which is annuitized. Naturally, in Table 4, we observe that the optimal fraction increases in the risk aversion with respect to consumption. Furthermore, we observe that the optimal fraction of wealth invested in the annuity decreases in the temporal risk aversion, as already suggested by Table 3.

Finally, Table 5 presents the recommended proportion of financial resources to allocate to the annuity, assuming a risk loading of 4%. This risk loading falls within the 2–5% range specified in the study by Huang, Milevsky, and Young (2017).

Comparing the values in Table 5 with the values provided in Table 4 for $\gamma = 3$, we observe that all values have drastically decreased. This decrease is a predictable outcome since the inclusion of a risk loading reduces annuitization rates and makes tontines relatively more appealing. Additionally, it is noteworthy that even under conditions of temporal risk neutrality, individuals exhibit a preference for investing positive fractions in tontines, aligning with the findings of, for example, Stamos (2008), Hanewald, Piggott, and Sherris (2013), Milevsky and Salisbury (2015) and Chen, Rach, and Sehner (2020).

3.3.1. Remark on Social Planner's Perspective

Our main analysis conducted a complementary analysis from the perspective of a utilitarian social planner. This analysis, which is available upon request, examines optimal consumption decisions across individuals with heterogeneous mortality profiles and wealth levels.

The key distinction from the individual case lies in the planner's objective to maximize the collective utility across groups, subject to an aggregate resource constraint. While the mathematical structure of the resulting optimization problems is similar, particularly regarding the forms of the optimal payout functions, the Lagrange multipliers differ, reflecting the aggregation of

TABLE 4

Sensitivity Analysis of the Fraction of Wealth Optimally Invested in the Annuity, with the Parameters Chosen as in Table 2 and $\delta_a = 0$

	$\theta = 0$	$\theta = 0.035$	$\theta = 0.07$	$\theta = 0.105$	$\theta = 0.14$
$\gamma = 0.5$	100%	15%	5%	1.5%	0.5%
$\gamma = 3$	100%	52%	39%	30%	25%
$\gamma = 6$	100%	65%	53%	45%	40%
$\gamma = 9$	100%	72%	62%	54%	50%

TABLE 5

Sensitivity Analysis of the Fraction of Wealth Optimally Invested in the Annuity, with the parameters chosen as in Table 2 and $\delta_a = 4\%$

$\theta = 0$	$\theta = 0.035$	$\theta = 0.07$	$\theta = 0.105$	$\theta = 0.14$
2.0%	1.7%	1.5%	1.3%	1.1%

multiple individuals' constraints into one. This changes the wealth allocation, particularly across groups with differing life expectancies.

Our findings confirm that individuals with longer expected lifespans tend to benefit from pooling with shorter-lived individuals, consistent with prior literature (e.g., Bommier, Leroux, and Lozachmeur 2011; Brown 2002). However, this redistribution can reverse if the short-lived group possesses significantly more initial wealth. Temporal risk aversion can either amplify or reduce these wealth transfers, depending on specific parameter configurations.

Importantly, the qualitative effects of temporal risk aversion observed in the individual case remain robust under the social planner framework: Individuals still favor earlier consumption under greater temporal risk aversion, and tontines become increasingly attractive. Thus, the analysis from the social planner's perspective validates and reinforces the core insights of our individual-based analysis, without altering the main conclusions.

4. CONCLUSION

This article examines the optimal demand for retirement products in the context of temporal risk aversion within a market that offers both traditional annuities and mortality-linked products such as tontines. We explore the optimal retirement decisions in scenarios involving an exclusive annuity market, an exclusive tontine market, and a market comprising both types of products.

Our theoretical and numerical results suggest that agents' demand for tontines increases under temporal risk aversion. While it has been shown that no tontinization and full annuitization is optimal in the absence of risk loadings (e.g., Chen, Rach, and Sehner 2020), we determine conditions for pricing bounds under which agents subject to temporal risk aversion prefer to invest positive fractions of wealth in both annuities and tontines. In particular, we find that temporal risk aversion leads individuals to seek a diversification strategy throughout retirement, which results in high consumption if mortality rates are high and low consumption if mortality rates are low. This can be captured by tontinizing some wealth, while at the same time annuitizing the remaining wealth to ensure the security of guaranteed retirement benefits. Most importantly, we find that tontines and annuities do not crowd each other out in the optimum, meaning that agents subject to temporal risk aversion prefer partial annuitization along with partial tontinization to full annuitization and full tontinization.

As an extension of this study, the inclusion of products providing unit-linked retirement benefits in the market could provide valuable insights. It would be interesting to investigate whether these products have the potential to outperform annuities and/or tontines in the context of temporal risk aversion.

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DISCLOSURE STATEMENT

No potential conflict of interest was reported by the authors.

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APPENDIX A. TECHNICAL APPENDIX

A.1. Approximating the Expected Lifetime Utility

First, the expected lifetime utility is given by

$$\begin{aligned}
 EU\left(\left\{\frac{d(t)}{S_x(t)}\right\}_{t \geq 0}\right) &= \mathbb{E}\left[\Phi\left(\int_0^{T_x} \alpha(t) u\left(\frac{d(t)}{S_x(t)}\right) dt\right)\right] \\
 &= \mathbb{E}\left[\mathbb{E}\left[\Phi\left(\int_0^{T_x} \alpha(t) u\left(\frac{d(t)}{S_x(t)}\right) dt\right) \middle| \{\mu_{x+t}\}_{t \geq 0}\right]\right] \\
 &= \mathbb{E}\left[\int_0^\infty \mu_{x+t} \cdot S_x(t) \cdot \Phi\left(\int_0^t \alpha(s) u\left(\frac{d(s)}{S(s)}\right) ds\right) dt\right].
 \end{aligned} \tag{16}$$

By integration by parts and using [Assumption 2.1](#), we rearrange [Equation \(16\)](#) to

$$EU\left(\left\{\frac{d(t)}{S_x(t)}\right\}_{t \geq 0}\right) = \mathbb{E}\left[\int_0^\infty S_x(t) \cdot \Phi'\left(\int_0^t \alpha(s) u\left(\frac{d(s)}{S(s)}\right) ds\right) \alpha(t) u\left(\frac{d(t)}{S_x(t)}\right) dt\right]. \tag{17}$$

We can now follow similar steps as in [Appendix A.1](#) in Bommier, Leroux, and Lozachmeur (2011). In particular, we assume a “priceless life context” as defined in Bommier (2006). In other words, we assume that $u(y) = 1 + \psi w(y)$, where ψ is a constant close to zero and the function $w(\cdot)$ is bounded. Then, we can rewrite [Equation \(17\)](#) to

$$\begin{aligned}
 EU\left(\left\{\frac{d(t)}{S_x(t)}\right\}_{t \geq 0}\right) &= \mathbb{E}\left[\int_0^\infty S_x(t) \cdot \Phi'\left(\int_0^t \alpha(s) \left(1 + \psi w\left(\frac{d(s)}{S(s)}\right)\right) ds\right) \alpha(t) dt\right. \\
 &\quad \left.+ \psi \int_0^\infty S_x(t) \cdot \Phi'\left(\int_0^t \alpha(s) \left(1 + \psi w\left(\frac{d(s)}{S(s)}\right)\right) ds\right) \alpha(t) w\left(\frac{d(t)}{S_x(t)}\right) dt\right].
 \end{aligned}$$

Next, we assume that agents would accept giving up most of their consumption to live longer, which translates into $\psi \rightarrow 0$. We can then approximate the function Φ' using a Taylor expansion. Note that Taylor series have been frequently used to approximate expected utility (e.g., Levy and Markowitz 1979; Hlawitschka 1994; Markowitz 2014). Ignoring the terms of order greater than 1 in ψ , this delivers

$$\begin{aligned}
EU\left(\left\{\frac{d(t)}{S_x(t)}\right\}_{t \geq 0}\right) &\approx \mathbb{E}\left[\int_0^\infty S_x(t) \cdot \Phi'\left(\int_0^t \alpha(s) ds\right) \alpha(t) dt \right. \\
&\quad + \psi \int_0^\infty S_x(t) \cdot \Phi'\left(\int_0^t \alpha(s) ds\right) \alpha(t) w\left(\frac{d(t)}{S_x(t)}\right) dt \\
&\quad \left. + \psi \int_0^\infty S_x(t) \cdot \left(\int_0^t \alpha(s) w\left(\frac{d(s)}{S(s)}\right) ds\right) \Phi''\left(\int_0^t \alpha(s) ds\right) \alpha(t) dt\right].
\end{aligned}$$

Here, we change the order of integration in the third term to obtain the following approximation of the expected lifetime utility:

$$\begin{aligned}
EU\left(\left\{\frac{d(t)}{S_x(t)}\right\}_{t \geq 0}\right) &\approx \mathbb{E}\left[\int_0^\infty S_x(t) \cdot \Phi'\left(\int_0^t \alpha(s) ds\right) \alpha(t) dt \right. \\
&\quad + \psi \int_0^\infty S_x(t) \cdot \Phi'\left(\int_0^t \alpha(s) ds\right) \alpha(t) w\left(\frac{d(t)}{S_x(t)}\right) dt \\
&\quad \left. + \psi \int_0^\infty w\left(\frac{d(t)}{S_x(t)}\right) \cdot \left(\int_t^\infty \alpha(s) S(s) \Phi''\left(\int_0^s \alpha(\tau) d\tau\right) ds\right) \alpha(t) dt\right] \\
&= \mathbb{E}\left[\int_0^\infty S_x(t) \cdot \Phi'\left(\int_0^t \alpha(s) ds\right) \alpha(t) dt + \psi \int_0^\infty w\left(\frac{d(t)}{S_x(t)}\right) \alpha(t) \right. \\
&\quad \left. \cdot \left(S_x(t) \cdot \Phi'\left(\int_0^t \alpha(s) ds\right) + \int_t^\infty \alpha(s) S(s) \Phi''\left(\int_0^s \alpha(\tau) d\tau\right) ds\right) dt\right].
\end{aligned}$$

Applying the integration by parts technique, we obtain

$$S_x(t) \cdot \Phi'\left(\int_0^t \alpha(s) ds\right) + \int_t^\infty \alpha(s) S(s) \Phi''\left(\int_0^s \alpha(\tau) d\tau\right) ds = - \int_t^\infty S'_x(s) \Phi'\left(\int_0^t \alpha(s) ds\right) ds.$$

This delivers the following approximation for the expected lifetime utility:

$$EU\left(\left\{\frac{d(t)}{S_x(t)}\right\}_{t \geq 0}\right) \approx \mathbb{E}\left[\int_0^\infty S_x(t) \Phi'\left(\int_0^t \alpha(s) ds\right) \alpha(t) dt + \psi \int_0^\infty w\left(\frac{d(t)}{S_x(t)}\right) \alpha(t) S_x(t) \beta(t) dt\right],$$

where $\beta(t)$ is defined as in [Equation \(5\)](#). Using again $u(y) = 1 + \psi w(y)$ delivers

$$\begin{aligned}
EU\left(\left\{\frac{d(t)}{S_x(t)}\right\}_{t \geq 0}\right) &\approx \mathbb{E}\left[\int_0^\infty S_x(t) \Phi'\left(\int_0^t \alpha(s) ds\right) \alpha(t) dt \right. \\
&\quad \left. + \int_0^\infty u\left(\frac{d(t)}{S_x(t)}\right) \alpha(t) S_x(t) \beta(t) dt - \int_0^\infty \alpha(t) S_x(t) \beta(t) dt\right].
\end{aligned}$$

Dropping the constants in this expression, we end up with the additive lifetime utility function

$$EU\left(\left\{\frac{d(t)}{S_x(t)}\right\}_{t \geq 0}\right) \approx \mathbb{E}\left[\int_0^\infty u\left(\frac{d(t)}{S_x(t)}\right) \alpha(t) S_x(t) \beta(t) dt\right],$$

which is equal to [\(4\)](#).

A.2. Proof of Theorem 3.3

The first-order condition is given by

$$\mathbb{E}[S_x(t)^\gamma \beta(t)] \alpha(t) u'(d(t)) = (1 + \delta_d) \lambda_d B(t).$$

Hence, we obtain the optimal payout function (7). Plugging Equation (7) back into the budget constraint delivers

$$\begin{aligned} W_0 &= (1 + \delta_d) \int_0^\infty B(t) \cdot \left(\frac{(1 + \delta_d) \lambda_d B(t)}{\mathbb{E}[S_x(t)^\gamma \beta(t)] \alpha(t)} \right)^{-\frac{1}{\gamma}} dt \\ &\iff (\lambda_d)^{\frac{1}{\gamma}} = \frac{1 + \delta_d}{W_0} \int_0^\infty B(t) \cdot \left(\frac{(1 + \delta_d) B(t)}{\mathbb{E}[S_x(t)^\gamma \beta(t)] \alpha(t)} \right)^{-\frac{1}{\gamma}} dt. \end{aligned}$$

For $\gamma \neq 1$, the expected discounted lifetime utility is then given by

$$\begin{aligned} \int_0^\infty \mathbb{E}[S_x(t)^\gamma \beta(t)] \alpha(t) u(d^*(t)) dt &= \int_0^\infty \mathbb{E}[S_x(t)^\gamma \beta(t)] \alpha(t) u \left(\left(\frac{(1 + \delta_d) \lambda_d B(t)}{\mathbb{E}[S_x(t)^\gamma \beta(t)] \alpha(t)} \right)^{-\frac{1}{\gamma}} \right) dt \\ &= \frac{(\lambda_d)^{1-\frac{1}{\gamma}}}{1-\gamma} \int_0^\infty (1 + \delta_d) B(t) \left(\frac{(1 + \delta_d) B(t)}{\mathbb{E}[S_x(t)^\gamma \beta(t)] \alpha(t)} \right)^{-\frac{1}{\gamma}} dt \\ &= \frac{(\lambda_d)^{1-\frac{1}{\gamma}}}{1-\gamma} \cdot (\lambda_d)^{\frac{1}{\gamma}} \cdot W_0 \\ &= \frac{\lambda_d}{1-\gamma} \cdot W_0. \end{aligned}$$

□

A.3. Lemma A.1

Lemma A.1. *It holds $\mathbb{E}[S_x(t) \beta(t)] = s_x(t) \bar{\beta}(t)$.*

Proof. The equality can be obtained by the following transformations:

$$\begin{aligned} \mathbb{E}[S_x(t) \beta(t)] &= \mathbb{E} \left[- \int_t^\infty S'_x(s) \cdot \Phi' \left(\int_0^s \alpha(\tau) d\tau \right) ds \right] \\ &= - \int_t^\infty \mathbb{E}[S'_x(s)] \cdot \Phi' \left(\int_0^s \alpha(\tau) d\tau \right) ds \\ &= - \int_t^\infty s'(s) \cdot \Phi' \left(\int_0^s \alpha(\tau) d\tau \right) ds \\ &= \bar{\beta}(t) s_x(t). \end{aligned}$$

□

A.4. Proof of Theorem 3.4

The first-order condition is given by

$$\bar{\beta}(t) \alpha(t) u'(c(t)) = (1 + \delta_a) \lambda_a B(t).$$

Hence, we obtain the solution to problem (11) as Equation (12). Plugging the optimal consumption (12) back into the budget constraint delivers

$$\begin{aligned} W_0 &= (1 + \delta_a) \int_0^\infty B(t) \cdot s_x(t) \cdot \left(\frac{(1 + \delta_a)\lambda_a B(t)}{\bar{\beta}(t)\alpha(t)} \right)^{-\frac{1}{\gamma}} dt \\ &\iff (\lambda_a)^{\frac{1}{\gamma}} = \frac{1 + \delta_a}{W_0} \int_0^\infty B(t) \cdot s_x(t) \cdot \left(\frac{(1 + \delta_a)B(t)}{\bar{\beta}(t)\alpha(t)} \right)^{-\frac{1}{\gamma}} dt. \end{aligned}$$

For $\gamma \neq 1$, the expected discounted lifetime utility is then given by

$$\begin{aligned} \int_0^\infty s_x(t) \bar{\beta}(t) \alpha(t) u(c(t)) dt &= \int_0^\infty s_x(t) \bar{\beta}(t) \alpha(t) u \left(\left(\frac{(1 + \delta_a)\lambda_a B(t)}{\bar{\beta}(t)\alpha(t)} \right)^{-\frac{1}{\gamma}} \right) dt \\ &= \frac{(\lambda_a)^{1-\frac{1}{\gamma}}}{1-\gamma} \int_0^\infty (1 + \delta_a) B(t) s_x(t) \left(\frac{(1 + \delta_a)B(t)}{\bar{\beta}(t)\alpha(t)} \right)^{-\frac{1}{\gamma}} dt \\ &= \frac{(\lambda_a)^{1-\frac{1}{\gamma}}}{1-\gamma} \cdot (\lambda_a)^{\frac{1}{\gamma}} \cdot W_0 \\ &= \frac{\lambda_a}{1-\gamma} \cdot W_0. \end{aligned}$$

□

A.5. Proof of Theorem 3.5

The proof can be carried out in a similar way as in Chen, Rach, and Sehner (2020).

1. The objective function to problem (13) can be written as

$$\begin{aligned} \mathcal{F} &= \int_0^\infty \mathbb{E} \left[\alpha(t) S_x(t) \beta(t) u \left(c_{ad}(t) + \frac{d_{ad}(t)}{S_x(t)} \right) \right] dt \\ &\quad + \lambda (W_0 - (1 + \delta_a) \int_0^\infty B(t) s_x(t) c_{ad}(t) dt - (1 + \delta_d) \int_0^\infty B(t) \cdot d_{ad}(t) dt). \end{aligned}$$

Setting the first-order conditions equal to zero, we obtain

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial c_{ad}(t)} &= \mathbb{E} \left[\alpha(t) S_x(t) \beta(t) u' \left(c_{ad}(t) + \frac{d_{ad}(t)}{S_x(t)} \right) \right] - \lambda (1 + \delta_a) B(t) s_x(t) = 0 \\ \frac{\partial \mathcal{F}}{\partial d_{ad}(t)} &= \mathbb{E} \left[\alpha(t) \beta(t) u' \left(c_{ad}(t) + \frac{d_{ad}(t)}{S_x(t)} \right) \right] - \lambda (1 + \delta_d) B(t) = 0. \end{aligned}$$

Now we set $c_{ad}(t) = c^*(t)$, $d_{ad}(t) = 0$ and $\lambda = \lambda_a$ to obtain a condition under which the annuity is optimal, using Lemma A.1:

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial c_{ad}(t)} &= \mathbb{E} [\alpha(t) S_x(t) \beta(t) u'(c^*(t))] - \lambda_a (1 + \delta_a) B(t) s_x(t) \\ &= \lambda_a (1 + \delta_a) s_x(t) B(t) - \lambda_a (1 + \delta_a) B(t) s_x(t) = 0. \end{aligned}$$

Further, we obtain

$$\begin{aligned}
\frac{\partial \mathcal{F}}{\partial d_{ad}(t)} &= \mathbb{E}[\alpha(t)\beta(t)u'(c^*(t))] - \lambda_a(1 + \delta_d)B(t) \\
&= \mathbb{E}\left[\frac{\beta(t)}{\beta(t)}\lambda_a(1 + \delta_d)B(t)\right] - \lambda_a(1 + \delta_d)B(t) \\
&= \lambda_a B(t) \left((1 + \delta_d) \mathbb{E}\left[\frac{\beta(t)}{\beta(t)}\right] - (1 + \delta_d) \right) \stackrel{!}{\leq} 0.
\end{aligned}$$

Now let us take a look at the second-order derivative:

$$\frac{\partial^2 \mathcal{F}}{\partial d_{ad}(t)^2} = \mathbb{E}\left[\alpha(t)\beta(t)u''\left(c_{ad}(t) + \frac{d_{ad}(t)}{S_x(t)}\right) \frac{1}{S_x(t)}\right] < 0$$

since $u'' < 0$. If the second-order derivative is strictly negative, this implies that the first-order derivative is strictly decreasing in $d_{ad}(t)$. If $\partial \mathcal{F}(\alpha)/\partial d_{ad}(t) < 0$, the expected lifetime utility could be increased by choosing the payout function $d_{ad}(t)$ even lower, which is not possible because we exclude negative retirement benefits. If $\partial \mathcal{F}(\alpha)/\partial d_{ad}(t) = 0$, $d_{ad}(t) = 0$ is the optimal solution and, therefore, the annuity is the optimal retirement product. Therefore, the annuity is optimal if and only if

$$(1 + \delta_a) \mathbb{E}\left[\frac{\beta(t)}{\beta(t)}\right] \leq (1 + \delta_d) \quad \text{for all } t \geq 0.$$

2. Now we set $c_{ad}(t) = 0$, $d_{ad}(t) = d^*(t)$ and $\lambda = \lambda_d$ to obtain a condition under which the annuity is optimal:

$$\begin{aligned}
\frac{\partial \mathcal{F}}{\partial c_{ad}(t)} &= \mathbb{E}\left[\alpha(t)S_x(t)\beta(t)u'\left(\frac{d^*(t)}{S_x(t)}\right)\right] - \lambda_d(1 + \delta_a)B(t)s_x(t) \\
&= \mathbb{E}\left[S_x(t)^{\gamma+1}\beta(t)\right]\alpha(t)u'(d^*(t)) - \lambda_d(1 + \delta_a)B(t)s_x(t) \\
&= \frac{\mathbb{E}\left[S_x(t)^{\gamma+1}\beta(t)\right]}{\mathbb{E}\left[S_x(t)^\gamma\beta(t)\right]}\lambda_d(1 + \delta_d)B(t) - \lambda_d(1 + \delta_a)B(t)s_x(t) \\
&= \lambda_d B(t) \left(\frac{\mathbb{E}\left[S_x(t)^{\gamma+1}\beta(t)\right]}{\mathbb{E}\left[S_x(t)^\gamma\beta(t)\right]}(1 + \delta_d) - (1 + \delta_a)s_x(t) \right).
\end{aligned}$$

Further, we obtain

$$\begin{aligned}
\frac{\partial \mathcal{F}}{\partial d_{ad}(t)} &= \mathbb{E}\left[\alpha(t)\beta(t)u'\left(\frac{d^*(t)}{S_x(t)}\right)\right] - \lambda_d(1 + \delta_d)B(t) \\
&= \alpha(t)\mathbb{E}\left[S_x(t)^\gamma\beta(t)\right]u'(d^*(t)) - \lambda_d(1 + \delta_d)B(t) \\
&= 0.
\end{aligned}$$

Now let us take a look at the second-order derivative:

$$\frac{\partial^2 \mathcal{F}}{\partial c_{ad}(t)^2} = \mathbb{E}\left[\alpha(t)S_x(t)\beta(t)u''\left(c_{ad}(t) + \frac{d_{ad}(t)}{S_x(t)}\right)\right] < 0$$

since $u'' < 0$. If the second-order derivative is strictly negative, this implies that the first-order derivative is strictly decreasing in $c_{ad}(t)$. If $\partial \mathcal{F}(x)/\partial c_{ad}(t) < 0$, the expected lifetime utility could be increased by choosing $c_{ad}(t)$ even lower, which is not possible because we exclude negative retirement benefits. If $\partial \mathcal{F}(x)/\partial c_{ad}(t) = 0$, $c_{ad}(t) = 0$ is the optimal solution and, therefore, the tontine is the optimal retirement product. Therefore, the tontine is optimal if and only if

$$\frac{\mathbb{E}[S_x(t)^{\gamma+1}\beta(t)]}{\mathbb{E}[S_x(t)^\gamma\beta(t)]}(1 + \delta_d) - (1 + \delta_a)s_x(t) \leq 0.$$

□

APPENDIX B. CALIBRATION OF BENCHMARK PARAMETERS IN TABLE 2

We calibrate our model in such a way that a reasonable set of benchmark parameters is obtained:

- **Initial wealth:** The initial wealth W_0 amounts to 100 and can simply be interpreted as 100%.
- **Temporal risk aversion:** Following Bommier, Leroux, and Lozachmeur (2011), we assume that Φ is an exponential utility function with constant absolute risk aversion $\theta > 0$ and choose θ in such a way that a plausible discounting rate is achieved.
- **Relative risk aversion:** Davies (1981) reports 3, 4, and 5 to be the most reasonable estimates of relative risk aversion. Additionally, Azar (2010) finds 3.01 and 3.74 as estimates for bounds of relative risk aversion. In particular, a relative risk aversion coefficient of 3 has also been frequently used in life cycle studies (cf. Cocco and Gomes 2012; Kim, Maurer, and Mitchell 2016).
- **Risk-free interest rate:** We assume a constant risk-free interest rate close to zero, based on slowly increasing positive interest rates in many countries. For example, in Germany, the risk-free interest rate in 2022 was 1.2% (Statista 2023).
- **Subjective discount factor:** The subjective discount factor $\alpha(t)$ is equal to the risk-free interest rate, which allows us to focus exclusively on the effects of temporal risk aversion as an additional subjective discount factor. Furthermore, observe that the one-year subjective discount factor is $e^{-0.01} \approx 0.99$ for any $t \geq 0$ close to previous life cycle studies (e.g., Cocco and Gomes 2012).
- **Risk loading:** Following Stamos (2008), Hanewald, Piggott, and Sherris (2013), and Milevsky and Salisbury (2015), we assume that the risk loading of the tontine is equal to zero. This assumption can be justified by the observation that tontine providers do not carry any significant mortality risk. The annuity, however, may come with a risk loading.
- **Retirement age:** We choose 65 years as a retirement age as this is the standard retirement age in many countries. This age has, for example, also been used in Peijnenburg, Nijman, and Werker (2016).
- **Gompertz mortality law:** For the Gompertz parameters m and b , we follow Milevsky and Salisbury (2015), resulting in a probability of approximately 5% to reach age 100. We fix a maximum age of 115.
- **Mortality shock:** Concerning the mortality shock ε , we follow Chen, Hieber, and Klein (2019) and choose a truncated normal distribution on the interval $(-\infty, 1)$. In particular, the parameters of the truncated normal distribution are chosen in such a way that the longevity shock of the Solvency II standard formula occurs with a probability of 0.5%.

APPENDIX C. ON THE ACCURACY OF APPROXIMATION (4)

Throughout this article, we rely on the approximations of the generalized life-cycle utility function of an economic agent. These approximations can be seen as an extension to Bommier, Leroux, and Lozachmeur (2011), who have derived a similar approximation for annuities in a model with no systematic mortality risk. In particular, Bommier, Leroux, and Lozachmeur (2011) have already studied the accuracy of their approximation for annuities and have found that it performs relatively well. In order to avoid redundancy across the literature, we therefore only focus on the approximation of the lifetime utility resulting under tontines. To be precise, we are interested in determining how accurately the approximation (4) approximates the general life cycle utility function under temporal risk aversion (3).

In order to check the accuracy of this approximation, we determine the expected lifetime utility of two tontine structures under the approximation (4) and the formal representation (3). We consider the optimal tontine structure derived in Equation (7) and a structure that is called *flat* tontine, which is based on a constant payout function $d(t) \equiv d$. As this payout function has been criticized for its increasing retirement benefits over time (see Milevsky and Salisbury 2015), it is clear that this design will deliver a lower expected lifetime utility than the (approximately) optimal structure (7). Note that the

TABLE C.1
Ratios ($U_{\text{approx}}^{\text{optimal}}/U_{\text{approx}}^{\text{flat}}, U_{\text{theoretical}}^{\text{optimal}}/U_{\text{theoretical}}^{\text{flat}}$) Depending on the Risk Aversion with Respect to Consumption, with the Parameters Taken from Table 2

Risk aversion	$\gamma = 0.5$	$\gamma = 3$	$\gamma = 6$	$\gamma = 9$
Quotient	(1.220,1.073)	(0.328,0.342)	(0.048,0.052)	(0.0065,0.0072)

TABLE C.2
Ratios ($U_{\text{approx}}^{\text{optimal}}/U_{\text{approx}}^{\text{flat}}, U_{\text{theoretical}}^{\text{optimal}}/U_{\text{theoretical}}^{\text{flat}}$) Depending on the Temporal Risk Aversion, with the Parameters Taken from Table 2

Risk aversion	$\theta = 0.035$	$\theta = 0.07$	$\theta = 0.105$	$\theta = 0.14$
Quotient	(0.328,0.342)	(0.308,0.344)	(0.287,0.353)	(0.264,0.370)

magnitudes of the approximation (4) and the formal representation (3) differ because we drop constant terms in the derivation in Appendix A.1. Therefore, simply comparing the absolute values of expected utility of the two tontine structures does not reveal the degree of accuracy, which is why we determine the relative utility loss resulting from the flat tontine under both lifetime felicity functions. To be precise, let $U_{\text{approx}}^{\text{optimal}}, U_{\text{approx}}^{\text{flat}}, U_{\text{theoretical}}^{\text{optimal}}$, and $U_{\text{theoretical}}^{\text{flat}}$ be the respective utility functions under the corresponding lifetime felicity functions and tontine structures; then we compare $U_{\text{approx}}^{\text{optimal}}/U_{\text{approx}}^{\text{flat}}$ to $U_{\text{theoretical}}^{\text{optimal}}/U_{\text{theoretical}}^{\text{flat}}$. The closer the two ratios are, the better the approximation (4) performs.

In Table C.1, we present the resulting losses of utility under both utility functions for different degrees of risk aversion with respect to consumption.

Naturally, we observe that the ratios decrease in the risk aversion, because individuals with a higher relative risk aversion require more stable retirement benefits than provided by the flat tontine. Additionally, and more importantly for this section, the two ratios are relatively close to each other and the magnitude of the loss resulting from the suboptimal flat design coincides for the different degrees of risk aversion. In other words, the results suggest that the approximation works relatively fine for different degrees of relative risk aversion with respect to consumption.

In Table C.2, we present the resulting losses of utility under both utility functions for different degrees of temporal risk aversion.

We observe that the approximate values are relatively close to the theoretical values, although the difference increases as the degree of temporal risk aversion increases. Hence, we should be careful not to apply the approximation to any degree of temporal risk aversion and keep in mind that the approximation (and with it the results in our article) is based on the assumption of an extremely high value of life (compared to consumption).